

# VARIETIES CONNECTED BY CHAINS OF LINES

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**ABSTRACT.** In this paper we give for any integer  $l \geq 2$  a numerical criterion ensuring the existence of a chain of length  $l$  of lines through two general points of an irreducible variety  $X \subset \mathbb{P}^N$ , involving the degrees and the number of homogeneous polynomials defining  $X$ . We show that our criterion is sharp.

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## INTRODUCTION

*Bonavero* and *Höring*, in [BH], consider a smooth scheme theoretical complete intersection  $X \subset \mathbb{P}^N$  and give a bound involving the degrees of the polynomials defining the variety that grants the conic-connectedness. However their result ensures the existence of a smooth conic that in general is weaker than the existence of a singular conic. Indeed from classical arguments of deformations of chains of rational curves we have that a singular conic through two general points on a smooth variety can be deformed into a smooth conic. The existence of a smooth conic  $f : \mathbb{P}^1 \rightarrow X$  through two general points on a projective variety does not imply the existence of a singular conic. This is true if  $\dim_{[f]}(\text{Mor}(\mathbb{P}^1, X; f_{\{0, \infty\}})) \geq 2$  by Mori's Bend-and-Break lemma [De, Proposition 3.2].

We start from this result and consider the more general case of a non necessarily smooth variety  $X \subset \mathbb{P}^N$ , set theoretically defined by homogeneous polynomials; such varieties do not need to be a complete intersection. In [MMT, Theorem 4.4] the authors and *Saeed Tafazolian* give a numerical criterion ensuring the existence of a chain of length two of lines through two general points of a variety  $X \subset \mathbb{P}^N$ . In section 2 we generalize [MMT, Theorem 4.4] and we obtain the following result (Theorem 2.1).

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1991 *Mathematics Subject Classification.* 14M99, 14N05, 14J45, 14M07.

*Key words and phrases.* Conic-connected varieties, covered by lines varieties, rationally chain connected varieties.

**Theorem.** *Let  $X \subset \mathbb{P}^N$  be a variety set theoretically defined by homogeneous polynomials  $G_i$  of degree  $d_i$ , for  $i = 1, \dots, m$ , and let  $l \geq 2$  be an integer. If*

$$\sum_{i=1}^m d_i \leq \frac{N(l-1) + m}{l}$$

*then  $X$  is rationally chain connected by chains of lines of length at most  $l$ . In particular if  $X$  is smooth and the above inequality is satisfied then  $X$  is rationally connected by rational curves of degree at most  $l$ .*

Finally we prove the sharpness of this result considering a hypersurface  $X_{l+1}$  of degree  $l+1$  in  $\mathbb{P}^{l+2}$ .

## 1. NOTATION AND PRELIMINARIES

We work over the complex field. Throughout this paper we denote by  $X \subset \mathbb{P}^N$  an irreducible variety of dimension  $n \geq 1$ . We assume  $X$  to be non-degenerate of codimension  $c$ , so that  $N = n + c$ .

*Prime Fano, covered by lines and conic connected varieties.* Let  $x \in X \subset \mathbb{P}^N$  be a general point. We denote by  $\mathcal{L}_x$  the (possibly empty) variety of lines through  $x$ , contained in  $X$ . Note that  $\mathcal{L}_x$  is embedded in the space of tangent directions at  $x$ , that is  $\mathcal{L}_x \subseteq \mathbb{P}(t_x X^*) = \mathbb{P}^{n-1}$ , where  $t_x X$  denotes the affine embedded Zariski tangent space at  $x$ .

We denote by  $a := \dim(\mathcal{L}_x)$  the dimension of  $\mathcal{L}_x$ . We say that  $X$  is *covered by lines* if  $\mathcal{L}_x \neq \emptyset$  for  $x \in X$  general. When  $\mathcal{L}_x$  is irreducible, it can be proved that  $a = \deg(\mathcal{N}_{l/X})$ , where  $l$  is a line in  $X$  through  $x$ , and  $\mathcal{N}_{l/X}$  is its normal bundle. When  $a \geq \frac{n-1}{2}$ ,  $\mathcal{L}_x \subset \mathbb{P}^{n-1}$  is smooth and irreducible; if, moreover,  $\text{Pic}(X)$  is cyclic, it is also non-degenerate, see [Hw].

Recall that  $X \subset \mathbb{P}^N$  is a *prime Fano variety of index  $i(X)$*  if its Picard group is generated by the class  $H$  of a hyperplane section and  $-K_X = i(X)H$  for some positive integer  $i(X)$ . By the work of Mori, see [Mo], if  $i(X) > \frac{n+1}{2}$  then  $X$  is covered by lines.

A variety  $X \subset \mathbb{P}^N$  is called a *conic-connected (CC) variety* if for  $x, y \in X$  general points there is a conic  $C_{x,y}$  passing through  $x, y$  and contained in  $X$ .

*Loci of Chains.* Let  $X$  be a variety covered by lines and let  $x \in X$  be a general point. We define the loci determined on  $X$  by chains of lines through  $x$  as follows.

**Definition 1.1.** The locus of lines in  $X$  through  $x$  is defined as

$$\mathfrak{Loc}_1(x) = \bigcup_{[L] \in \mathcal{L}_x} L$$

and the locus of chains of lines of length  $l$  in  $X$  through  $x$  is defined recursively as

$$\mathfrak{Loc}_l(x) = \overline{\bigcup_{[L] \in \mathcal{L}_y \mid y \in \mathfrak{Loc}_{l-1}(x) \text{ is a general point}} L}.$$

We denote by  $d_l$  the maximal dimension of the irreducible components of  $\mathfrak{Loc}_l(x)$ . If there exists an integer  $l$  such that  $d_l = \dim(X)$  but  $d_{l-1} < \dim(X)$  we say that  $X$  has *length  $l$* . In particular a variety of length 2 is conic-connected.

In [HK] *Hwang* and *Kebekus* give a lower bound on  $d_l$  under the irreducibility assumption on  $\mathcal{L}_x$  for  $x \in X$  general point. However their proof work even without this irreducibility assumption. The following theorem can be found in [Wa, Theorem 4.6].

**Theorem 1.2.** *Let  $X$  be a prime Fano variety of dimension  $n \geq 3$ . Then we have*

$$d_1 = \dim(\mathcal{L}_x) + 1 \text{ and } d_l \leq l(\dim(\mathcal{L}_x) + 1) \text{ for each } l \geq 1.$$

In the proof of Theorem 2.1 it will be necessary to perform intersections in a product of projective spaces, so we recall briefly the structure of the Chow ring of a product, for more details see [Fu].

*Chow ring of a product.* Let us recall, in order to fix the notation, that for a cartesian product of projective spaces the Chow ring is given by

$$A^*(\mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_k}) \cong \mathbb{Z}[h_1, \dots, h_k] / (h_1^{N_1+1}, \dots, h_k^{N_k+1})$$

where  $h_i$  is the hyperplane class of  $\mathbb{P}^{N_i}$ . It follows that the class of subvariety  $Z \subset \mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_k}$  can be written in the form

$$[Z] = \sum_{i_1 + \dots + i_k = \dim(Z)} \lambda_{i_1, \dots, i_k} h_1^{N_1 - i_1} \dots h_k^{N_k - i_k},$$

where the  $\lambda_{i_1, \dots, i_k}$  are the multidegrees of  $Z$ .

## 2. CHAINS OF LINES

We want to generalize [MMT, Theorem 4.4] giving conditions on the equations defining the variety that ensure the existence of a chain of lines of a prescribed length through two general points of  $X$ . In this case we have to perform a intersection in a product of projective spaces.

**Theorem 2.1.** *Let  $X \subset \mathbb{P}^N$  be a variety set theoretically defined by homogeneous polynomials  $G_i$  of degree  $d_i$ , for  $i = 1, \dots, m$ , and let  $l \geq 2$  be an integer. If*

$$\sum_{i=1}^m d_i \leq \frac{N(l-1) + m}{l}$$

*then  $X$  is rationally chain connected by chains of lines of length at most  $l$ .*

*In particular if  $X$  is smooth and the above inequality is satisfied then  $X$  is rationally connected by rational curves of degree at most  $l$ .*

*Proof.* Let  $x, y \in X$  be two general points. We can assume that  $x = [1 : 0 : \dots : 0]$  and  $y = [0 : \dots : 0 : 1]$ . Moreover let us consider other  $l-1$  points in  $\mathbb{P}^N$ , that we denote by  $p^i = [p_0^i : \dots : p_N^i]$ , for  $i = 1, \dots, l-1$ . Our goal is to find the set of  $l-1$  points that will represent the intersections of the lines we are seeking to build the chain and we would like to see such set as a point of the cartesian product  $\mathbb{P}_1^N \times \dots \times \mathbb{P}_{l-1}^N$ .

Let us consider the line that join the points  $x$  and  $p^1$ , that we will denote by  $ux + vp^1 = [u + vp_0^1 : \dots : vp_N^1]$ . We may observe that  $G_i(ux + vp^1)$  is a polynomial of degree  $d_i$  in the variables  $u$  and  $v$ ; it has  $d_i + 1$  coefficients, but in this case there is no monomial of the type  $u^{d_i}$  because of our assumption  $x \in X$ . So imposing  $G_i(ux + vp^1) \equiv 0$  gives us  $d_i$  conditions, involving only on the coordinates of the point  $p^1$ .

Let us now consider the line  $up^1 + vp^2 = [up_0^1 + vp_0^2 : \dots : up_N^1 + vp_N^2]$ . Again

$G_i(up^1 + vp^2)$  is a homogeneous polynomial of degree  $d_i$  in the variables  $u, v$  and imposing  $G_i(up^1 + vp^2) \equiv 0$  gives us  $d_i + 1$  equations. Let us give a closer look to such equations;  $d_i - 1$  of them will be homogeneous polynomials in the coordinates of the points  $p^1, p^2$ , one of them will be a polynomial only in coordinates of the point  $p^1$ , which is exactly the equation  $G_i = 0$  written in the coordinates of  $p^1$  that we have already found in the previous step, and one of them will be a polynomial only in coordinates of the point  $p^2$ .

In the same way for every line  $up^{i-1} + vp^i = [up_0^{i-1} + vp_0^i : \dots : up_N^{i-1} + vp_N^i]$ , for  $i$  from 2 to  $l - 1$ , imposing  $G_i(up^{i-1} + vp^i) \equiv 0$ , we get  $d_i + 1$  equations;  $d_i - 1$  of them will be homogeneous polynomials in the coordinates of the points  $p^{i-1}, p^i$ , one of them will be a polynomial only in the coordinates of the point  $p^{i-1}$ , which is the equation  $G_i = 0$  written in the coordinates of  $p^{i-1}$  that we have already found in the previous step, and one of them will be a polynomial only in coordinates of the point  $p^i$ .

We now consider the line  $up^{l-1} + vy = [up_0^{l-1} : \dots : up_N^{l-1} + v]$ ; we notice that  $G_i(up^{l-1} + vy)$  is a polynomial of degree  $d_i$  in the variables  $u$  and  $v$ , it has  $d_i + 1$  coefficients, but in this case there is no monomial of the type  $v^{d_i}$  because of our assumption  $y \in X$ . So imposing  $G_i(up^{l-1} + vy) \equiv 0$  gives us  $d_i$  conditions, only on the coordinates of the point  $p^{l-1}$ .

Summarizing, we get the following conditions:

- $\sum_{i=1}^m d_i$  homogeneous equations only in the  $p_j^1$ 's and  $\sum_{i=1}^m d_i$  equations only in the  $p_j^{l-1}$ 's,
- $m$  homogeneous equations only in the  $p_j^k$ 's, for every  $k = 2, \dots, l - 2$ ,
- $\sum_{i=1}^m d_i - m$  bihomogeneous equations in the variables  $p_j^{k-1}, p_j^k$ 's, for every  $k = 2, \dots, l - 1$ .

We want to perform the intersection of these hypersurfaces in  $\mathbb{P}_1^N \times \dots \times \mathbb{P}_{l-1}^N$ . If  $h_i$  is the hyperplane class of  $\mathbb{P}_i^N$  the intersection is given by an expression of the form

$$h_1^{\sum_{i=1}^m d_i} h_2^m \dots h_{l-2}^m h_{l-1}^{\sum_{i=1}^m d_i} (h_1 + h_2)^{\sum_{i=1}^m d_i - m} \dots (h_{l-2} + h_{l-1})^{\sum_{i=1}^m d_i - m},$$

and each summand of the expression above is of the form

$$\begin{aligned} & h_1^{\sum_{i=1}^m d_i} h_2^m \dots h_{l-2}^m (h_1^{j_1} h_2^{\sum_{i=1}^m d_i - m - j_1}) (h_2^{j_2} h_3^{\sum_{i=1}^m d_i - m - j_2}) \dots (h_{l-2}^{j_{l-2}} h_{l-1}^{\sum_{i=1}^m d_i - m - j_{l-2}}) \\ & = h_1^{\sum_{i=1}^m d_i + j_1} h_2^{\sum_{i=1}^m d_i - j_1 + j_2} h_3^{\sum_{i=1}^m d_i - j_2 + j_3} \dots h_{l-2}^{\sum_{i=1}^m d_i - j_{l-3} + j_{l-2}} h_{l-1}^{2 \sum_{i=1}^m d_i - m - j_{l-2}}. \end{aligned}$$

Our aim is to prove that under the numerical hypothesis of the theorem at least one of these summands does not vanish. Take

$$\overline{j_k} = \lfloor \frac{k}{l-1} (\sum_{i=1}^m d_i - m) \rfloor, \quad \text{for } k = 1, \dots, l - 2,$$

where  $\lfloor p \rfloor$  is the greatest integer smaller or equal than  $p$ .

Note that  $0 \leq \overline{j_k} \leq \sum_{i=1}^m d_i - m$  for  $k = 1, \dots, l - 2$ . Consider the term

$$h_1^{\sum_{i=1}^m d_i + \overline{j_1}} h_2^{\sum_{i=1}^m d_i - \overline{j_1} + \overline{j_2}} h_3^{\sum_{i=1}^m d_i - \overline{j_2} + \overline{j_3}} \dots h_{l-2}^{\sum_{i=1}^m d_i - \overline{j_{l-3}} + \overline{j_{l-2}}} h_{l-1}^{2 \sum_{i=1}^m d_i - m - \overline{j_{l-2}}}.$$

For any  $k = 2, \dots, l - 2$  the exponent of  $h_k$  is  $\sum_{i=1}^m d_i - \overline{j_{k-1}} + \overline{j_k}$ . In order to ensure the intersection in  $\mathbb{P}_k^N$  to be not empty we impose

$$N - \sum_{i=1}^m d_i + \overline{j_{k-1}} - \overline{j_k} \geq 0 \text{ for any } k = 2, \dots, l - 2.$$

Substituting we have

$$\lfloor \frac{k-1}{l-1} (\sum_{i=1}^m d_i - m) \rfloor \geq \lfloor \frac{k}{l-1} (\sum_{i=1}^m d_i - m) \rfloor - N + \sum_{i=1}^m d_i.$$

Since the number on the right is an integer it is enough to prove that  $\frac{k-1}{l-1} (\sum_{i=1}^m d_i - m) \geq \lfloor \frac{k}{l-1} (\sum_{i=1}^m d_i - m) \rfloor - N + \sum_{i=1}^m d_i$  that is

$$\frac{k-1}{l-1} (\sum_{i=1}^m d_i - m) + N - \sum_{i=1}^m d_i \geq \lfloor \frac{k}{l-1} (\sum_{i=1}^m d_i - m) \rfloor.$$

Consider now the term on the left, from our hypothesis we get  $N \geq \frac{l \sum_{i=1}^m d_i - m}{l-1}$ . So

$$\begin{aligned} \frac{k-1}{l-1} (\sum_{i=1}^m d_i - m) + N - \sum_{i=1}^m d_i &\geq \frac{k-1}{l-1} (\sum_{i=1}^m d_i - m) + \frac{l \sum_{i=1}^m d_i - m}{l-1} - \sum_{i=1}^m d_i \\ &= \frac{k}{l-1} (\sum_{i=1}^m d_i - m) \geq \lfloor \frac{k}{l-1} (\sum_{i=1}^m d_i - m) \rfloor. \end{aligned}$$

This prove that the intersection in  $\mathbb{P}_k^N$  is not empty for any  $k = 2, \dots, l-2$ .

Consider now the exponent of  $h_1$ . We have to impose

$$N - \sum_{i=1}^m d_i - \bar{j}_1 = N - \sum_{i=1}^m d_i - \lfloor \frac{\sum_{i=1}^m d_i - m}{l-1} \rfloor \geq 0.$$

But  $N - \sum_{i=1}^m d_i \geq \frac{\sum_{i=1}^m d_i - m}{l-1}$  implies the last inequality and is equivalent to the numerical hypothesis of the theorem. This show that the intersection in  $\mathbb{P}_1^N$  is also not empty.

Finally consider the exponent of  $h_{l-1}$  and impose

$$N - 2 \sum_{i=1}^m d_i + m + \bar{j}_{l-2} \geq 0.$$

We have  $\lfloor \frac{l-2}{l-1} (\sum_{i=1}^m d_i - m) \rfloor \geq 2 \sum_{i=1}^m d_i - m - N$ , since the number on the right is an integer it is enough to prove that  $(l-2)(\sum_{i=1}^m d_i - m) \geq (l-1)(2 \sum_{i=1}^m d_i - m - N)$  and again this is exactly our numerical hypothesis. We conclude that the intersection in  $\mathbb{P}_{l-1}^N$  is not empty.

At this point we know that the equations define non-empty subvarieties of  $\mathbb{P}_j^N$  for any  $j = 1, \dots, l-1$ . To ensure that these subvarieties lift to subvarieties of  $\mathbb{P}_1^N \times \dots \times \mathbb{P}_{l-1}^N$  rationally equivalent to cycles having non-empty intersection we force

$$N(l-1) - (l-2) \sum_{i=1}^m d_i - 2 \sum_{i=1}^m d_i + m \geq 0,$$

which once again is our initial hypothesis.

So our system of equations on the product  $\mathbb{P}_1^N \times \dots \times \mathbb{P}_{l-1}^N$  has at least a solution, which represents the sequence of connection points in the chain of lines we were looking for. Finally, if  $X$  is smooth, by general smoothing arguments ([De], Proposition 4.24) we can deform our chain in a rational curve of degree at most  $l$  connecting  $x$  and  $y$ .  $\square$

**2.1. Sharpness of Theorem 2.1.** The inequality in Theorem 2.1 is sharp. Consider a smooth hypersurface  $X_{l+1}$  of degree  $l+1$  in  $\mathbb{P}^{l+2}$ . Then  $d = l+1$ ,  $N = l+2$  and  $m = 1$ , so we have  $\frac{N(l-1)+m}{l} = \frac{l^2+l-1}{l}$ . Since

$$l \leq \frac{l^2+l-1}{l} < l+1$$

the hypersurface  $X_{l+1}$  is a good candidate to prove sharpness. The equalities imply  $d = l+1 = l+2-1 = N-1$  we have  $\dim(\mathcal{L}_x) = 0$ . Now by Theorem 1.2 the dimension  $d_l = \dim(\mathcal{L}\mathcal{O}_l(x))$  is bounded by  $d_l \leq l(\dim(\mathcal{L}_x) + 1) = l$ . Since  $\dim(X_{l+1}) = l+1 > l$  we have

$$\dim(\mathcal{L}\mathcal{O}_l(x)) < \dim(X)$$

and  $X$  is not connected by chains of length  $l$  of lines.

From Theorem 2.1 we get the following Corollary which can also be found in [Ko, Lemma 4.8.1].

**Corollary 2.2.** *Let  $X \subset \mathbb{P}^N$  be a hypersurface of degree  $d \leq N-1$ . Then  $X$  is rationally chain connected by a chain of lines of length at most  $N-1$ .*

*Proof.* If  $l = N-1$  the inequality  $d \leq \frac{N(N-2)+1}{N-1} = \frac{N^2-2N+1}{N-1} = N-1$  of Theorem 2.1 is satisfied by hypothesis.  $\square$

**Corollary 2.3.** *Let  $X \subset \mathbb{P}^N$  be a scheme theoretical complete intersection. If  $\deg(X) \leq \frac{N(l-1)+c}{l}$  then  $X$  is rationally chain connected by chains of lines of length at most  $l$ . If, in addition,  $X$  is smooth and Fano of index  $i_X \geq \frac{n+l}{l}$  then  $X$  is rationally connected by rational curves of degree at most  $l$ .*

*Proof.* The first assertion follows from the inequality  $\sum_{i=1}^c d_i \leq \prod_{i=1}^c d_i = \deg(X)$  combined with Theorem 2.1. If  $X$  is a smooth, Fano, complete intersection, Theorem 2.1 and the equality  $i_X = N+1 - \sum_{i=1}^c d_i$  imply the second assertion.  $\square$

**Proposition 2.4.** *Let  $X \subseteq \mathbb{P}^N$  be a smooth complete intersection defined by homogeneous polynomials  $G_i$  of degree  $d_i$ , for  $i = 1, \dots, c$ , such that  $\sum_{i=1}^c d_i \leq N-1$ . Then*

$$\text{length}(X) = \lceil \frac{N-c}{N - \sum_{i=1}^c d_i} \rceil,$$

where  $\lceil k \rceil$  is the smallest integer greater or equal than  $k$ .

*Proof.* Since the integer  $l_{\min} := \lceil \frac{N-c}{N - \sum_{i=1}^c d_i} \rceil$  satisfies the inequality of Theorem 2.1 we have that  $X$  is  $l_{\min}$ -chain connected. We have to prove that  $X$  is not  $(l_{\min} - 1)$ -chain connected. Now  $l_{\min} - 1 = \lceil \frac{N-c}{N - \sum_{i=1}^c d_i} \rceil - 1$ . Note that  $\dim(\mathcal{L}_x) = N - \sum_{i=1}^c d_i - 1 \geq 0$ . By Theorem 1.2 we have  $d_{l_{\min}-1} \leq (l_{\min} - 1)(\dim(\mathcal{L}_x) + 1)$ , we distinguish two cases

- If  $\frac{N-c}{N - \sum_{i=1}^c d_i} - 1$  is an integer then

$$d_{l_{\min}-1} \leq \left( \frac{N-c}{N - \sum_{i=1}^c d_i} - 1 \right) (N - \sum_{i=1}^c d_i) = \sum_{i=1}^c d_i - c < N - c = n.$$

- If  $\frac{N-c}{N-\sum_{i=1}^c d_i} - 1$  is not an integer then

$$d_{l_{\min}-1} \leq (l_{\min} - 1)(N - \sum_{i=1}^c d_i) < \frac{N-c}{N-\sum_{i=1}^c d_i}(N - \sum_{i=1}^c d_i) = n.$$

Then  $d_{l_{\min}-1} = \dim(\mathfrak{Loc}_{l_{\min}-1}(x)) < \dim(X)$ .  $\square$

**Remark 2.5.** In the case  $l = 2$  we find again [MMT, Theorem 4.4], in fact, the inequality in 2.1 simply becomes  $\sum_{i=1}^m d_i \leq \frac{N+m}{2}$ .

**Remark 2.6.** In the range of Theorem 2.1  $X$  is covered by lines. Indeed under the numerical hypothesis of Theorem 2.1 we have  $m < \sum_{i=1}^m d_i \leq \frac{N(l-1)+m}{l}$  which gives  $m < N$ . So we get the inequality

$$\sum_{i=1}^m d_i \leq \frac{N(l-1)+m}{l} < N,$$

which forces  $X$  to be covered by lines.

*Counting chains of lines.* We discuss now an example that shows how it is possible to count the number of possible chains of lines when the equality in Theorem 2.1 holds.

Let us consider a cubic threefold  $X \subset \mathbb{P}^4$ . In this case the equality holds when  $l = 3$ , so we are looking for all the possible 3-chains of lines connecting two general points  $x, y$  of  $X$ . Following the proof of the theorem, we have to perform intersection in  $\mathbb{P}_1^4 \times \mathbb{P}_2^4$ , we are looking for two points  $p^1, p^2$ . We have 3 conditions on the coordinates of  $p^1$  namely  $h_1, 2h_1, 3h_1$ , describing the cone of lines in  $X$  through  $x$ . Furthermore we have 3 other conditions  $h_2, 2h_2, 3h_2$  on the coordinates of  $p^2$ , describing the cone of lines in  $X$  through  $y$ . Finally we have 2 conditions involving the coordinates of both points  $p^1, p^2$ , namely  $2h_1 + h_2$  and  $h_1 + 2h_2$ . Their intersection product is given by

$$h_1 2h_1 3h_1 h_2 2h_2 3h_2 (2h_1 + h_2)(h_1 + 2h_2) = 36h_1^3 h_2^3 (2h_1^2 + 5h_1 h_2 + 2h_2^2) = 180h_1^4 h_2^4,$$

and we conclude that we have 180 possibilities.

A geometrical description of this fact is the following: there are exactly  $6 = h_1 2h_1 3h_1$  lines in  $X$  through  $x$  and  $6 = h_2 2h_2 3h_2$  lines in  $X$  through  $y$ . Take a line  $L_x$  of the first family and a line  $L_y$  of the second. These lines are skew otherwise  $X$  would be connected by singular conics and we know this is not possible by classical arguments of projective geometry, so  $L_x$  and  $L_y$  generate a 3-plane  $H$ . The linear section  $H \cap X := S$  is a smooth cubic surface in  $\mathbb{P}^3$  and we can consider the lines  $L_x, L_y$  as two exceptional divisors of a proper blow-up of the projective plane in 6 points; we denote by  $p, q$  the points in the plane which are blown-up in  $L_x$  and  $L_y$ . There are exactly 5 lines joining  $L_x$  and  $L_y$  namely the strict transform of the line  $\langle p, q \rangle$  and of the conics passing through  $p, q$  and 3 of the 4 remaining points. In conclusion we have  $6 \cdot 6 \cdot 5 = 180$  possibilities, that double-checks the counting made before.

*Acknowledgements.* The authors heartily thank *Prof. Paltin Ionescu* and *Dr. José Carlos Sierra* for the introduction to the subject, many helpful comments and suggestions. We would like to give a special thank also to *Prof. Enrique Arrondo* and *Prof. Massimiliano Mella*.

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